



TITLE:

A representation of unital completely positive maps (Mathematical Studies on Independence and Dependence Structure : Algebra meets Probability)

AUTHOR(S):

Choda, Marie

CITATION:

Choda, Marie. A representation of unital completely positive maps (Mathematical Studies on Independence and Dependence Structure : Algebra meets Probability). 数理解析研究所講究録 2012, 1820: 11-24

ISSUE DATE:

2012-12

URL:

<http://hdl.handle.net/2433/194653>

RIGHT:

A representation of unital completely positive maps

大阪教育大学 長田 まり糸 (Marie Choda)

Abstract

Let $M_n(\mathbb{C})$ be the algebra of $n \times n$ complex matrices, and let Φ be a unital completely positive map of $M_n(\mathbb{C})$ to $M_k(\mathbb{C})$. With the notion of the von Neumann entropy for a state in mind, we give a model of r -tuple $\{v_j\}_{j=1}^r$ so that $\Phi(x) = v_1^* x v_1 + \cdots + v_r^* x v_r$, ($x \in M_n(\mathbb{C})$). The r is uniquely determined for Φ and the r -tuple is also unique up to a $r \times r$ unitary matrix.

1 Introduction

In the framework of the theory of operator algebras, the notion of entropy for automorphisms was introduced by Connes-Størmer in [8], Connes-Narnhofer-Thirring in [9] and Voiculescu in [14] (which is extended by Brown [4]). The Connes-Størmer entropy $H(\theta)$ is defined for a $*$ -automorphism θ of finite von Neumann algebra M with $\tau = \tau \circ \theta$, where τ is a fixed given finite trace of M . After then, the Connes-Narnhofer-Thirring entropy $h_\phi(\theta)$ is given as an extended version of $H(\theta)$ for a $*$ -automorphism θ of a C^* -algebra A by replacing the trace τ to a state ϕ of A , and if A is a finite von Neumann algebra then $h_\tau(\theta) = H(\theta)$. Voiculescu's topological entropy $ht(\theta)$ is defined as an independent version of any state of A .

We studied these entropies in [6] and [7] for not only $*$ -automorphisms but also $*$ -endomorphisms like so called canonical shifts. As one of interesting such $*$ -endomorphisms, we picked up the Cuntz canonical endomorphism Φ_n on the Cuntz algebra O_n which has a strong connection to Longo's canonical shift (cf. [6]). The O_n is the C^* -algebra generated by isometries $\{S_1, \dots, S_n\}$ such that $S_1 S_1^* + \cdots + S_n S_n^* = 1$, and the Φ_n is defined as

$$\Phi_n(x) = S_1 x S_1^* + \cdots + S_n x S_n^*, \quad (x \in O_n). \quad (1.1)$$

Such maps given by the form as the right hand side of (1.1) are unital completely positive maps, and the above notions $H(\cdot)$, $h_\phi(\cdot)$ and $ht(\cdot)$ are available for unital completely positive maps too.

Conditional expectations are the most typical examples of unital completely positive maps, and states of the matrix algebras $M_n(\mathbb{C})$ are considered as the most elementary example of conditional expectations. However, for a conditional expectation E , by their definitions it holds always that $H(E) = h_\phi(E) = ht(E) = 0$. On the other hand, in the case of the von Neumann entropy $S(\phi)$ for a state ϕ of $M_n(\mathbb{C})$, it is possible that $S(\phi) \neq 0$.

In order to define "entropy", we need the notion of "finite partition of unity" (see for example, [11]). The most generalized one of "finite partition of unity" was introduced by Lindblad ([10]), and it is called the "finite operational partition of unity".

With these facts in mind, here we give a method to induce the finite operational partition of unity for a given unital completely positive map. That is, let A and B be unital C^* -algebras and let Φ be a unital completely positive map of A to B . We give a method to get a model of r -tuple $v(\Phi) = \{v_1, v_2, \dots, v_r\}$ such that

$$\Phi(x) = v_1^* x v_1 + \dots + v_r^* x v_r, \quad (x \in A). \quad (1.2)$$

When A and B are matrix algebras, such a representation is called Kraus representation (cf. Appendix in [13]), or obtained as a straightforward application of Stinespring's theorem (see for example, [1, 3]). We note that this representation is not unique

Our main purpose in this note is to show, for a given completely positive map Φ , a unique r -tuple $v(\Phi) = \{v_1, \dots, v_r\}$ which is suitable to extend the notion of von Neumann entropy $S(\phi)$ for a state ϕ of matrix algebras to the entropy $S(\Phi)$ for a unital completely positive map Φ .

First, for a given completely positive map Φ from $B(H)$ to $B(K)$ of finite dimensional Hilbert spaces H, K , we construct the Hilbert spaces $H \otimes_\Phi K$. Let $r = \dim(H \otimes_\Phi K)$. Next, we give a r -tuple $v(\Phi) = \{v_1, v_2, \dots, v_r\}$ which satisfy that $\Phi(x) = v_1^* x v_1 + \dots + v_r^* x v_r$. The r -tuple is unique up to unitaries and induces $S(\Phi)$. After then, we apply these to the non-commutative Bernoulli shift β and we define the entropy $S_\varphi(\Phi)$ with respect to a state φ with $\varphi = \varphi \cdot \beta$. These results in this note are in [5].

2 Preliminaries

Here, we denote some notations and terminologies which we use later.

We denote by $M_n(\mathbb{C})$ the algebra of $n \times n$ complex matrices, and by Tr_n the standard trace, that is, the sum of all diagonal components. A matrix $D \in M_n(\mathbb{C})$ is called a *density matrix* if D is a positive operator with $\text{Tr}_n(D) = 1$ (cf. [11] [12]).

The notation η is called the entropy function in usual, and it is the function defined by

$$\eta(t) = \begin{cases} -t \log t, & (0 < t \leq 1) \\ 0, & t = 0 \end{cases}$$

2.1 Finite partitions

The notion of "a finite partition of unity" is the starting point of our study.

2.1.1 Finite partitions of 1

The first one is discussed in the real numbers \mathbb{R} . Let

$$\lambda = \{\lambda_1, \dots, \lambda_n\}$$

be the set of real numbers $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$. We say that the n -tuple $\lambda \subset \mathbb{R}$ is a *finite partition of 1*.

2.1.2 Finite operational partition of unity

The terminology, a *finite operational partition of unity*, was first given by Lindblad ([10]) and after then it is used by Alicki-Fannes([2]).

Let A be a unital C^* -algebra. Let $x = \{x_1, \dots, x_k\} \subset A$. Then x is said to be a *finite operational partition of unity* of size k if

$$\sum_i^k x_i^* x_i = 1_A. \quad (2.1)$$

Such a finite operational partition of unity $x = \{x_1, \dots, x_k\}$ in A induces an A -coefficient in $M_k(A)$, whose (i, j) coefficient $x(j, i)$ is given by the following:

$$x(j, i) = x_i^* x_j, \quad (1 \leq i, j \leq k). \quad (2.2)$$

We denote this matrix by $[x]$. Then $[x]$ is an A -coefficient density matrix in $M_k(A)$, that is, $[x]$ is a positive operator with $\text{Tr}([x]) = \sum_{i=1}^k x(i, i) = 1_A$.

2.2 Entropy for finite partitions of unity

2.2.1 Entropy for finite partitions of 1

Let a n -tuple $\lambda \subset \mathbb{R}$ be a *finite partition of 1*. Let

$$H(\lambda) = \eta(\lambda_1) + \cdots + \eta(\lambda_n). \quad (2.3)$$

Then $H(\lambda)$ is called the *entropy* for the finite partition λ of 1.

2.2.2 Entropy for Finite operational partition of unity

Let $x = \{x_1, \dots, x_k\}$ be an operational partition of unity in a unital C^* -algebra A , and let φ be a state of A . The $\rho_\varphi[x]$ is the $k \times k$ matrix whose $\{i, j\}$ -component is defined by

$$\rho_\varphi[x](i, j) = \varphi(x_j^* x_i), \quad (i, j = 1, \dots, k). \quad (2.4)$$

Then $\rho_\varphi[x]$ is a density matrix. We call $\rho_\varphi[x]$ the *density matrix associate with x and φ* . If φ is a unique tracial state, we denote $\rho_\varphi[x]$ by $\rho[x]$ simply.

Let $\lambda(\rho_\varphi[x]) = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ be the eigenvalues of the matrix $\rho_\varphi[x]$. Then $\lambda(\rho_\varphi[x])$ is a finite partition of 1 because $\rho_\varphi[x]$ is a density matrix. Hence we have the entropy $H(\lambda(\rho_\varphi[x]))$.

Let $S(\rho_\varphi[x])$ be the von Neumann entropy (cf. [11, 12]) for the density matrix $\rho_\varphi[x]$. Then $S(\rho_\varphi[x])$ is nothing else but $H(\lambda(\rho_\varphi[x]))$, that is,

$$S(\rho_\varphi[x]) = \text{Tr}_k(\eta(\rho[x])) = H(\lambda(\rho_\varphi[x])) = \sum_i \eta(\lambda_i). \quad (2.5)$$

3 Representation of completely positive maps

Let Φ be a completely positive map of $M_n(\mathbb{C})$ to $M_k(\mathbb{C})$. Put $A = M_n(\mathbb{C})$. We give a method to get a "finite" family $v(\Phi) = \{v_1, v_2, \dots, v_r\}$ for Φ which satisfies that $\Phi(x) = v_1^* x v_1 + \cdots + v_r^* x v_r$ for all $x \in A$. We remark that if Φ is unital, then $v(\Phi)$ is a finite operational partition:

$$\sum_{j=1}^r v_j^* v_j = \Phi(1_A) = 1_{B(K)} \quad (3.1)$$

3.1 Hilbert space $H \otimes_{\Phi} K$

Let H be an n -dimensional Hilbert space, and let $\Phi : A \rightarrow B(K)$ be a completely positive linear map. Let $\{e_1, \dots, e_m\}$ be the set of mutually orthogonal minimal projections in $B(H)$ with $\Phi(e_i) \neq 0$ for all i . Let $\xi_i \in e_i(H)$ be a vector with $\|\xi_i\| = 1$ for $i = 1, \dots, m$ and we extend $\{\xi_1, \dots, \xi_m\}$ to an orthonormal basis of H as $\{\xi_1, \dots, \xi_n\}$. Let $\{e_{ij}; i, j = 1, \dots, n\}$ be a matrix units of A with $e_{ij}\xi_j = \xi_i$ so that $e_{ii} = e_i$ for $i = 1, \dots, m$. Then each $\zeta \in H \odot K$ (the algebraic tensor product $H \odot K$ of H and K) is written by

$$\zeta = \sum_{i=1}^n \xi_i \otimes \mu_i, \quad \text{for some } \mu_i \in K. \quad (3.2)$$

Definition 3.1.1. We define a sesquilinear form $\langle \cdot, \cdot \rangle_{\Phi}$ on the space $H \odot K$ by

$$\langle \sum_{i=1}^n \xi_i \otimes \mu_i, \sum_{j=1}^n \xi_j \otimes \nu_j \rangle_{\Phi} = \sum_{i,j} \langle \Phi(e_{ji})\mu_i, \nu_j \rangle_K, \quad (3.3)$$

where $\langle \cdot, \cdot \rangle_K$ means the inner product of the Hilbert space K .

Since Φ is completely positive, this form $\langle \cdot, \cdot \rangle_{\Phi}$ turns out positive semidefinite. The value $\langle \cdot, \cdot \rangle_{\Phi}$ depends on the choice of the orthonormal basis of H . However the kernel of this sesquilinear form $\langle \cdot, \cdot \rangle_{\Phi}$ is unique up to unitaries on $H \otimes K$ as follows:

Proposition 3.1.2. Let $\{\xi_1, \dots, \xi_n\}$ (resp., $\{\xi'_1, \dots, \xi'_n\}$) be an orthonormal basis of H , and let u be a unitary on $H \otimes K$ with $\xi'_i = u\xi_i$ for all $i = 1, \dots, n$. Let $\text{Ker}(\Phi)$ (resp., $\text{Ker}'(\Phi)$) be the kernel of this form via $\{\xi_i\}_i$ (resp., $\{\xi'_i\}_i$) Then

$$\text{Ker}'(\Phi) = (\bar{u}u^* \otimes 1)\text{Ker}(\Phi)$$

where \bar{u} is the unitary matrix on H whose (i, j) -entry is the conjugate complex number of $u(i, j)$ for all $i, j = 1, \dots, n$.

Definition 3.1.3. Now taking the quotient by the space $\text{Ker}(\Phi)$, we have a preHilbert space and complete to get the Hilbert space $H \otimes_{\Phi} K$.

We denote by $(\sum_{i=1}^n \xi_i \otimes \mu_i)_{\Phi}$ the element in $H \otimes_{\Phi} K$ corresponding to $\sum_{i=1}^n \xi_i \otimes \mu_i \in H \odot K$.

If K is finite dimensional, of course the $H \otimes_{\Phi} K$ is finite dimensional. We can extend this method to get the Hilbert space $H \otimes_{\Phi} K$ to (for an example) non-commutative Bernoulli shifts, and it induces finite dimensional $H \otimes_{\Phi} K$ even if H and K are infinite dimensional. By Proposition 3.1.2, we have the following:

Proposition 3.1.4. *The dimension of $H \otimes_{\Phi} K$ does not depend on the choice of orthonormal basis of H .*

Example 3.1.5.

1. If ϕ is a state of $M_n(\mathbb{C})$, then

$$\dim(\mathbb{C}^n \otimes_{\phi} \mathbb{C}) = \text{rank of } \phi, \text{ (i.e., the rank of the density matrix of } \phi).$$

2. If E is the conditional expectation of $M_n(\mathbb{C})$ to a maximal abelian subalgebra A of $M_n(\mathbb{C})$, then

$$\dim(\mathbb{C}^n \otimes_E \mathbb{C}^n) = n.$$

Here, we remark that A is isomorphic to the diagonal subalgebra $D_n(\mathbb{C})$.

3. Let B be a subfactor of $M_n(\mathbb{C})$. If E is the conditional expectation $M_n(\mathbb{C})$ to B , then

$$\dim(\mathbb{C}^n \otimes_E \mathbb{C}^m) = \frac{n}{m}$$

Here, we remark that B is isomorphic to $M_m(\mathbb{C})$ for some m by which n can be divided.

4. If α is an automorphism of $M_n(\mathbb{C})$, then

$$\dim(\mathbb{C}^n \otimes_{\alpha} \mathbb{C}^n) = 1.$$

The following shows that $\dim(H \otimes_{\Phi} K)$ can be finite, even if H and K are infinite dimensional.

Example 3.1.6. Let β be the non-commutative Bernoulli shift of $A = \bigotimes_{i=1}^{\infty} M_n(\mathbb{C})$. That is, let $A_i = M_n(\mathbb{C})$ for all $i = 1, 2, \dots$, and for each $m \in \mathbb{N}$, let

$$A(m) = A_1 \otimes \cdots \otimes A_m \otimes 1 \otimes \cdots \subset A, \quad (3.4)$$

where 1 is the unit of $M_n(\mathbb{C})$.

The β is given as the shift as the followings:

$$\beta(x) = 1 \otimes x \otimes 1 \otimes \cdots, \quad \text{for all } m, x \in A(m). \quad (3.5)$$

Let $H_i = \mathbb{C}^n$ for all $i = 1, 2, \dots$, and for each $m \in \mathbb{N}$. Fix an vector $\Omega \in \mathbb{C}$ with $\|\Omega\| = 1$, and let

$$H(m) = H_1 \otimes \cdots \otimes H_m \otimes \Omega \otimes \cdots \subset H = \otimes_{i=1}^{\infty} H_i, \quad (3.6)$$

The β is a unital completely positive map from $A \subset B(H)$ to $B(H)$, and the restriction $\beta|_{A(m)}$ of β to $A(m)$ is a unital completely positive map from $A(m) \subset B(H(m))$ to $B(H(m+1))$. Apply the above method to $\beta|_{A(m)}$, we have always that

$$\dim(H(m) \bigotimes_{\beta|_{A(m)}} H(m+1)) = n, \quad \text{for all } m$$

As a result, we have that

$$\dim(H \bigotimes_{\beta} H) = n = \dim(H \bigotimes_E H),$$

where $E : A \rightarrow \beta(A)$ is the conditional expectation.

Now, we call the dimension of $H \otimes_{\Phi} K$ the *rank* of Φ .

A phenomenon As an example, we show a phenomenon of the above discussion in the case of a state of $M_2(\mathbb{C})$ which indicates how the dimension of $H \otimes_{\phi} K$ coincides with the rank of a state ϕ of the usual sense.

Example 3.1.7. Let $\{\xi_1, \xi_2\}$ be an orthonormal basis of \mathbb{C}^2 and let $\{e_{ij}; i, j = 1, 2\}$ be a matrix units of \mathbb{C}^2 with $e_{ij}\xi_j = \xi_i$. We give a vector representation for each $\xi \in \mathbb{C}^2$ relative to this $\{\xi_i; i = 1, 2\}$ and a matrix representation for each $x \in M_2(\mathbb{C})$ relative to this $\{e_{ij}; i, j = 1, 2\}$:

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.7)$$

and

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (3.8)$$

Assume that ϕ is a state given by

$$\phi(x) = \phi\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = \frac{1}{2} \sum_{i,j=1}^2 x_{ij}. \quad (3.9)$$

Then $\phi(e_i) = 1/2$ for $i = 1, 2$ and the discussion with respect to $\{\xi_1, \xi_2\}$ and $\{e_{ij}; i, j = 1, 2\}$ is as follows:

Let Ω be a fixed unit vector in \mathbb{C} and let

$$\zeta_i = \sqrt{2}\xi_i \otimes \Omega \in \mathbb{C}^2 \otimes_{\phi} \mathbb{C}. \quad (3.10)$$

Then

$$\langle \zeta_i, \zeta_j \rangle_{\phi} = 2\phi(e_{ij}) = 1, \quad \text{for all } i, j = 1, 2. \quad (3.11)$$

This implies that $\zeta_i \in \mathbb{C}^2 \otimes_{\phi} \mathbb{C}$ has norm 1 for $i = 1, 2$ and

$$\zeta_1 = \sqrt{2}\xi_1 \otimes \Omega = \sqrt{2}\xi_2 \otimes \Omega = \zeta_2. \quad (3.12)$$

Next we choose another family of minimal projections with $\phi(p) \neq 0$ and orthonormal basis of \mathbb{C}^2 . Let

$$e'_{11} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad (3.13)$$

then $\phi(e'_{11}) = 1$ so that the set containing e'_{11} of minimal projections with $\phi(\cdot) \neq 0$ is the one point set $\{e'_{11}\}$. The corresponding orthonormal basis of \mathbb{C}^2 and the corresponding matrix units are as follows:

$$\xi'_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \xi'_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (3.14)$$

and

$$e'_{11}, e'_{12} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad e'_{21} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad e'_{22} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (3.15)$$

Let

$$\zeta'_i = \frac{1}{\sqrt{2}} \xi'_i \otimes \Omega \in \mathbb{C}^2 \otimes_{\phi} \mathbb{C}, \quad (i = 1, 2). \quad (3.16)$$

Then

$$\langle \zeta'_1, \zeta'_1 \rangle_{\phi} = \frac{1}{2} \phi(e'_{11}) = 1, \quad (3.17)$$

and

$$\langle \zeta'_2, \zeta'_2 \rangle_\phi = \frac{1}{2} \phi(e'_{22}) = 0 \quad (3.18)$$

This means that

$$\mathbb{C}^2 \otimes_\phi \mathbb{C} = \mathbb{C} \zeta'_i = \xi'_i \otimes \mathbb{C}.$$

We remark that $\{\xi_1, \xi_2\}$ and $\{\xi'_1, \xi'_2\}$ are combined as $u\xi_i = \xi'_i, (i = 1, 2)$ by the unitary u :

$$u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (3.19)$$

Some relation to the Choi matrix. For a completely positive map Φ of $M_n(\mathbb{C})$, it is given the so called the Choi matrix C_Φ .

In the case of Φ is a state ϕ of $M_n(\mathbb{C})$, we have the following relation between the sesquilinear form $\langle \cdot, \cdot \rangle_\phi$ and the coefficient of C_ϕ :

$$\langle \xi_i \otimes \Omega, \xi_j \otimes \Omega \rangle_\phi = C_\phi(j, i) \quad (3.20)$$

where $\{\xi_1, \dots, \xi_n\}$ is a orthonormal basis of \mathbb{C}^n and Ω is 1 considered as the vector in \mathbb{C} .

3.2 Operators $\{v_j; j = 1, \dots, r\}$

As the above section, let $\Phi : B(H) \rightarrow B(K)$ be a completely positive linear map. Here, we assume that H and K be finite dimensional, and let $r = \dim(H \otimes_\Phi K)$.

Definition 3.2.1. Let $\{\xi_i; i = 1, \dots, n\}$ be an orthonormal basis of H , and let $\{\zeta_j; j = 1, \dots, r\}$ be an orthonormal basis of $H \otimes_\Phi K$. Define $v_j : K \rightarrow H$ by

$$v_j(\mu) = \sum_{i=1}^n \langle \xi_i \otimes \mu, \zeta_j \rangle_\Phi \xi_i, \quad (\mu \in K) \quad (3.21)$$

Proposition 3.2.2. Let $\{v_1, v_2, \dots, v_r\}$ be the tuple obtained by (3.17). Then

(i) They satisfies the desired following property:

$$\Phi(x) = v_1^* x v_1 + \dots + v_r^* x v_r, \quad (x \in B(H)) \quad (3.22)$$

(ii) $\|v_i\| \leq 1$, for all $i = 1, \dots, r$;

(iii) $\{v_1, v_2, \dots, v_r\}$ are linearly independent.

Proposition 3.2.3. *The tuple $\{v_1, v_2, \dots, v_r\}$ for unital completely positive map Φ satisfy the following convenient properties to compute the von Neumann type entropy $S(\Phi)$.*

1. *If E is a conditional expectation of $M_n(\mathbb{C})$ onto $D_n(\mathbb{C})$ then $\{v_j : 1 \leq j \leq r\}$ are mutually orthogonal minimal projections.*

2. *If Φ is a $*$ -homomorphism, then $\{v_j : 1 \leq j \leq r\}$ are isometries with*

$$v_i v_j^* = \delta_{ij} 1. \quad (3.23)$$

We remark that the following results are well known (see for example [1, 13]) so that our tuple $\{v_1, v_2, \dots, v_r\}$ for unital completely positive map Φ is unique up to a unitary matrix:

Proposition 3.2.4. *Let H and K be finite dimensional Hilbert spaces. Assume that $v = \{v_1, v_2, \dots, v_r\}$ and $w = \{w_1, \dots, w_r\}$ are two families of operators in $B(K, H)$. Then*

$$v_1^* x v_1 + \dots + v_r^* x v_r = w_1^* x w_1 + \dots + w_r^* x w_r, \quad \text{for all } x \in B(H)$$

if and only if there is a unitary matrix $[u(i, j)] \in M_r(\mathbb{C})$ such that

$$v_i = \sum_{j=1}^r u(i, j) w_j, \quad i = 1, \dots, r.$$

Example 3.2.5. Let β be the non-commutative Bernoulli shift on $A = \bigotimes_{i=1}^{\infty} M_n(\mathbb{C})$. The n -tuple $\{v_j\}_j$ of β are as follows. We use the same notation as Example 3.1.6. Let

$$W_m = \{\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m), \quad \alpha_j \in \{1, 2, \dots, n\}\}. \quad (3.24)$$

and let

$$\xi_\alpha = \xi_{\alpha_1} \otimes \xi_{\alpha_2} \otimes \dots \otimes \xi_{\alpha_m} \otimes \Omega \otimes \dots \in H(m) \quad (3.25)$$

Then $\{\xi_\alpha; \alpha \in W_m\}$ is an orthonormal basis of $H(m)$, and

$$\{\xi_\alpha \otimes_\beta (\xi_i \otimes \xi_\alpha); i = 1, 2, \dots, n\} \quad (3.26)$$

is an orthonormal basis of $H(m) \otimes_\beta H(m+1)$. Then our tuple for β is given by

$$v_j(\xi_i \otimes \xi_\beta) = \delta_{ij} \xi_\beta, \quad \text{for all } m, \beta \in W_m. \quad (3.27)$$

We remark that $\{v_j^*\}_j$ are isometries satisfying the Cuntz relation.

3.3 Relation to Stinespring's theorem

Let $\Phi : A \subset B(H) \rightarrow B(K)$ be a completely positive map, where H and K are finite dimensional. Let $\{v_j : K \rightarrow H\}_{j=1}^r$ be the tuple by (3.17). We denote by L the Hilbert space of r -direct sum of H , i.e., $H \oplus \cdots \oplus H$. Let

$$V(\xi) = (v_1\xi, \dots, v_r\xi) \in L, \quad \xi \in K \quad (3.28)$$

and let

$$\pi(x) = \begin{bmatrix} x & 0 & \cdots & 0 \\ : & x & : & 0 \\ : & : & : & : \\ 0 & 0 & \cdots & x \end{bmatrix}, \quad x \in A. \quad (3.29)$$

Then we have the followings:

1. The π is a representation of the C^* -algebra A on the Hilbert space L .
2. The property that $\sum_{i=1}^r v_i^* v_i = 1_K$ means that the operator V defined by (3.20) is an isometry from K to L .
3. The property $\Phi(x) = \sum_{j=1}^r v_j^* x v_j$ is written as

$$\Phi(x) = (v_1^*, \dots, v_r^*) \begin{bmatrix} x & 0 & \cdots & 0 \\ : & x & : & 0 \\ : & : & : & : \\ 0 & 0 & \cdots & x \end{bmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} = V^* \pi(x) V \quad (3.30)$$

4. These imply that (π, V) can be considered as the pair obtained by the Stinespring's representation.
5. The property that $\{v_j\}_j$ are linearly independent satisfies that (π, V) is the minimal pair in the sense of Arveson [1].

4 Entropy for unital completely positive maps

In this section, we devote an application to the notion of entropy. Let Φ be a unital completely positive map of a C^* -algebra $A \subset B(H)$ to $B \subset B(K)$ and let $v(\Phi) = \{v_1, v_2, \dots, v_r\}$ be the tuple for Φ obtained by (3.21). Then the tuple $v(\Phi)$ is a finite operational partition of unity in $B(K)$. Hence we can apply the discussion in the section 2.2.2 to unital completely positive map.

4.1 Case of a state ϕ of $M_n(\mathbb{C})$

First, we consider the case of a state ϕ of $M_n(\mathbb{C})$. Let ϕ be a state of $M_n(\mathbb{C})$, and let $v(\phi) = \{v_j : 1 \leq j \leq r\}$ be the tuple associated with ϕ defined by (3.20).

Let τ be the unique tracial state of $M_n(\mathbb{C})$, that is $\tau(x) = \text{Tr}(x)/n$ for all $x \in M_n(\mathbb{C})$. The density matrix $\tau[v(\phi)]$ is given by (2.4). Let $\{e_i\}_{i=1}^m$ be the mutually orthogonal minimal projections in $M_n(\mathbb{C})$ such that $\phi(e_i) \neq 0$. Then we see that

$$\tau[v(\phi)](i, j) = \delta_{ij} \sqrt{\phi(e_i)\phi(e_j)}. \quad (4.1)$$

It is clear that $\tau[v(\phi)]$ is a diagonal matrix, and the entropy $S(\tau[v(\phi)])$ in the section 2.2.2 is nothing else but the von Neumann entropy $S(\phi)$ of ϕ :

$$S(\tau[v(\phi)]) = \sum_{j=1}^r \eta(\phi(e_j)) = - \sum_{j=1}^r \phi(e_j) \log \phi(e_j) = S(\phi). \quad (4.2)$$

4.2 Entropy for unital completely positive maps

On the basis of the fact in the above section 4.1, we denote the $S(\rho[v(\Phi)])$ for a unital completely positive map $\Phi : A \rightarrow B$ by $S(\rho(\Phi))$, and in the case of the tracial state ρ we use the same notation $S(\Phi)$ simply.

Here, we show the case of A which has a unique tracial state τ and we number the values of typical examples of von Neumann type entropy $S(\Phi)$ for unital completely positive maps Φ .

1. If ϕ is a state of $M_n(\mathbb{C})$, then

$$S(\phi) = \sum_{j=1}^r \eta(\lambda_j) \quad (4.3)$$

where $\{\lambda_j\}$ are eigenvalues of ϕ .

2. If E is the conditional expectation of $M_n(\mathbb{C})$ to a maximal abelian subalgebra B then

$$S(E) = \log n. \quad (4.4)$$

Compare this fact to that $H(E) = ht(E) = 0$.

3. If E is the conditional expectation of $M_n(\mathbb{C})$ to a subfactor B then

$$S(E) = \log \frac{n}{k}. \quad (4.5)$$

Here we remark that a subfactor B of $M_n(\mathbb{C})$ is isomorphic to $M_k(\mathbb{C})$ some k , and that n is divisible by k . Compare this fact to that $H(E) = ht(E) = 0$.

4. If α is an automorphism of $M_n(\mathbb{C})$, then

$$S(\alpha) = 0 \quad (4.6)$$

and this coincides with the fact that $H(\alpha) = ht(\alpha) = 0$.

5. If β is the non-commutative Bernoulli shift on $\otimes_{i=1}^{\infty} M_n(\mathbb{C})$, then

$$S(\beta) = \log n \quad (4.7)$$

and this coincides with the fact that $H(\beta) = ht(\beta) = \log n$.

6. If Φ_n is the Cuntz's canonical shift on O_n , then

$$S_{\Psi}(\Phi_n) = \log n \quad (4.8)$$

and this coincides with the fact that $h_{\Psi}(\Phi_n) = ht(\Phi_n) = \log n$. Here Ψ is the state of O_n which is given by the left inverse of Φ .

References

- [1] W. Arveson, *Quantum channels that preserve entanglement*, arXiv:0801.2531v2.
- [2] R. Alicki and M. Fannes, *Defining Quantum Dynamical entropy*, Lett. Math. Phys., **32** (1994), 75–82.
- [3] F. Benatti, *Dynamics, Information and Complexity in Quantum Systems*, 2009, Springer-Verlag.
- [4] N. Brown *Topological entropy in exact C^* -algebras*, Math. Ann., **314** (1999), 347–367.

- [5] M. Choda, *Entropy for unital completely positive finite rank maps*, Preprint.
- [6] M. Choda, *Entropy of Cuntz's canonical endomorphism*, *Pacific J. Math.*, **190**(1999), 235-245.
- [7] M. Choda, *A C^* -dynamical entropy and applications to canonical endomorphisms*, *J. Funct. Anal.*, **173** (2000), 453-480.
- [8] A. Connes and E. Størmer, *Entropy of II_1 von Neumann algebras*, *Acta Math.*, **134** (1975), 289-306.
- [9] A. Connes, H. Narnhofer, W. Thrring *Dynamical entropy of C^* -algebras and von Neumann algebras*, *Comm. Math. Phys.*, **112**(1987), 691-719.
- [10] G. Lindblad, *Entropy, information and quantum measurements*, *Commun. Math. Phys.*, **33**, pp.111-119, 1973.
- [11] S. Neshveyev and E. Størmer, *Dynamical entropy in operator algebras*, Springer-Verlag, Berlin(2006).
- [12] M. Ohya and D. Petz, *Quantum Entropy and Its Use*, Springer, 2004.
- [13] D. Petz, *Quantum Information Theory and Quantum Statistics*, Springer, Berlin Heidelberg 2008.
- [14] D. Voiculescu, *Dynamical approximation entropies and topological entropy in operator algebras*, *Commun. Math. Phys.*, **170** (1995), 249-281.

marie@cc.osaka-kyoiku.ac.jp